REPORT 1183

SUPERSONIC FLOW PAST OSCILLATING AIRFOILS INCLUDING NONLINEAR THICKNESS EFFECTS [1]

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SUMMARY

A solution to second order in thickness is derived for harmonically oscillating two-dimensional airfoils in supersonic flow. For slow oscillations of an arbitrary profile, the result is found as a series including the third power of frequency. For arbitrary frequencies, the method of solution for any specific profile is indicated, and the explicit solution derived for a single wedge.

Nonlinear thickness effects are found generally to reduce the torsional damping, and so to enlarge the range of Mach numbers within which torsional instability is possible. This destabilizing effect varies only slightly with frequency in the range involved in dynamic stability analysis, but may reverse to a stabilizing effect at high flutter frequencies. Comparison with a previous solution exact in thickness suggests that nonlinear effects of higher than second order are practically negligible.

The analysis utilizes a smoothing technique that replaces the actual problem by one involving no kinked streamlines. This stratagem eliminates all consideration of shock waves from the analysis, yet yields the correct solution for problems that actually contain shock waves.

INTRODUCTION

As linearized supersonic-flow theory is increasingly applied to problems of unsteady motion of lifting wings, the results are sometimes advanced with the warning that they may be significantly affected by nonlinear effects of thickness. Such caution is justified because it is known that even for steady flow linearized theory is often inadequate for predicting the pitching moment—and prediction of moments is one of the main objectives of unsteady-flow theory. It may be anticipated that nonlinear effects will become increasingly important as the Mach number falls toward unity, particularly for slow oscillations.

In the present work the effects of thickness are determined for a harmonically oscillating two-dimensional airfoil by calculating the second-order solution. This is the counterpart for unsteady motion of the well-known steady-flow result of Busemann (ref. 1). First, for slow oscillations a solution is found for an airfoil of arbitrary profile. The result is given as a series that includes terms up to the third power of the frequency. Second, for arbitrarily high frequencies it is shown that a solution can be found for any specific airfoil, and the solution is carried out explicitly for a single wedge. Finally, comparison is made with a previous solution for the wedge that is exact with respect to thickness (refs. 2 and 3), in order to assess the effects of nonlinear terms of higher than second order.

Extensive use is made of a smoothing technique, which replaces the actual problem by one having no kinked streamlines. This stratagem, which has been used previously and may prove useful in future problems, eliminates all consideration of shock waves from the analysis. It, nevertheless, leads to the correct second-order solution for the actual problem, which does involve shock waves.

METHOD OF ANALYSIS

STATEMENT OF PROBLEM

Consider a sharp-nosed airfoil flying through still air at a uniform supersonic velocity and executing small harmonic oscillations. We shall be concerned with calculating the instantaneous pressure at the surface and, hence, the unsteady lift and pitching moment. If oscillations in the flight direction are neglected, a rigid airfoil possesses two degrees of freedom. The oscillation can therefore be regarded as compounded of a rotation (pitching) and a vertical translation (plunging), which are not generally in phase.

Although the iteration procedure to be employed yields a formal result for any Mach number greater than unity, the solution probably breaks down when the flow becomes sonic at any point. Since this occurs at a Mach number somewhat higher than that for bow-wave detachment, the upper and lower surfaces of the airfoil operate independently in the probable range of validity of the solution. It is therefore sufficient to consider only the half field of flow lying above the airfoil, and this viewpoint will be adopted henceforth.

It is convenient to seek a solution to second order in the airfoil thickness, but to only first order in the amplitude of oscillation. This is sufficient because second-order terms in the oscillation, although affecting local pressures, have no effect upon lift or moment since they are equal on the upper and lower surfaces. Then the pitching and plunging components of the oscillation can be treated separately, and the results superimposed. Furthermore, it is enough to consider only pitching about an arbitrary pivot, because the plunging case can be recovered by letting the pivot recede to infinity and the pitching amplitude tend to zero, their product remaining finite. Thus, from the point of view of an

[1] Supersedes NACA TN 2982, "Supersonic Flow Past Oscillating Airfoils Including Nonlinear Thickness Effects" by Milton D. Van Dyke, 1953.

observer moving with the mean speed of the airfoil (or testing it in a wind tunnel) the airfoil is exposed to a uniform supersonic stream and oscillating slightly about a fixed pivot (fig. 1).

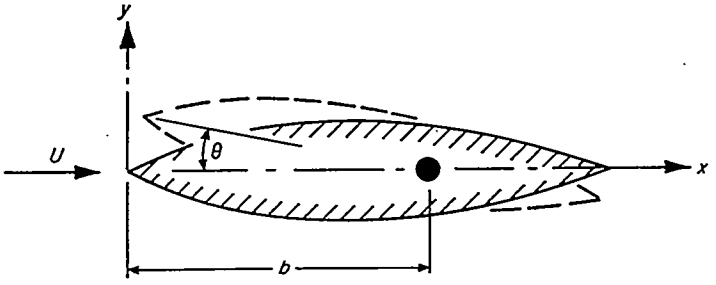


FIGURE 1.—Coordinate system for oscillating airfoil.

Choose the origin of coordinates at the mean position of the leading edge, with the x axis extending in the direction of the free stream. Then it is convenient to describe the upper surface of the airfoil in its mean (zero angle of pitch) position by

$$y=Y(x)=\epsilon f(x) \tag{1}$$

All symbols are defined in Appendix A. Here ϵ is a small parameter representative of the airfoil thickness, so that the function f is of order unity. Now let the airfoil pivot about a point on the x axis lying a distance b downstream from the leading edge, and perform harmonic oscillations of frequency ω and amplitude θ_0, so that the angle of pitch,[2] which is the angle between the instantaneous position of the airfoil and its original mean position (fig. 1) is given by

$$\theta=\theta_0 \cos \omega t=\theta_0 e^{i\omega t} \tag{2}$$

(Here, as in all that follows, it is implied that actual physical quantities are given by the real parts of their complex representations.) Then, at any instant the moving upper surface of the airfoil is described by

$$y=\epsilon f(x)-\theta_0 e^{i\omega t}(x-b) \tag{3}$$

with an error of order $(\epsilon^2\theta, \epsilon\theta^2)$, which is of third order and, consequently, negligible in the present second-order analysis.

PERTURBATION EQUATION

The entropy changes due to shock waves are of third order in the airfoil thickness and angle of pitch. Hence, to second order the flow is irrotational and isentropic. Because it is irrotational, there exists a potential function Ω whose gradient yields the velocity vector:

$$\vec{V}=\text{grad}\ \Omega, \quad \left.\begin{matrix} u=\Omega_x \\ v=\Omega_y \end{matrix}\right\} \tag{4}$$

Bernoulli's equation for plane unsteady flow can be written (from eqs. (14.04) and (9.06) of ref. 4)

$$\Omega_t+\frac{1}{2}(u^2+v^2)+\frac{a^2}{\gamma-1}=\frac{a_0^2}{\gamma-1}+\frac{1}{2}U^2 \tag{5}$$

Here a is the speed of sound, and a_0 its value in the free stream, where the flow velocity is U. Differentiating this expression with respect to time t, and using the fact that $d[a^2/(\gamma-1)]=a^2 d\rho/\rho$ (ref. 4, eqs. (9.03) and (9.06)) gives

$$\Omega_{tt}+u u_t+v v_t+\frac{a^2}{\rho}\rho_t=0 \tag{6}$$

This, together with the corresponding results obtained by differentiating with respect to x and y, can be used to eliminate derivatives of the density from the continuity equation (ref. 4, eq. (7.08.2))

$$\rho_t+(\rho u)_x+(\rho v)_y=0 \tag{7}$$

The result is that the velocity potential satisfies the equation

$$(a^2-\Omega_x^2)\Omega_{xx}+(a^2-\Omega_y^2)\Omega_{yy}-2\Omega_x\Omega_y\Omega_{xy}-2\Omega_x\Omega_{xt}-2\Omega_y\Omega_{yt}-\Omega_{tt}=0 \tag{8a}$$

where, from equations (4) and (5),

$$a^2=a_0^2+\frac{\gamma-1}{2}(U^2-\Omega_x^2-\Omega_y^2-2\Omega_t) \tag{8b}$$

Now introduce a perturbation potential Φ, normalized through division by the free-stream velocity U, by setting

$$\Omega(x, y, t)=U[x+\Phi(x, y, t)] \tag{9a}$$

so that the velocity components are given by

$$\left.\begin{matrix} \frac{u}{U}=1+\Phi_x \\ \frac{v}{U}=\Phi_y \end{matrix}\right\} \tag{9b}$$

Then substituting into equations (8) gives

$$(1-M^2)\Phi_{xx}+\Phi_{yy}-2\frac{M^2}{U}\Phi_{xt}-\frac{M^2}{U^2}\Phi_{tt}=M^2\left[(\gamma-1)\left(\Phi_x+\frac{\Phi_t}{U}+\frac{\Phi_x^2+\Phi_y^2}{2}\right)(\Phi_{xx}+\Phi_{yy})+(2\Phi_x+\Phi_x^2)\Phi_{xx}+\Phi_y^2\Phi_{yy}+2(1+\Phi_x)\Phi_y\Phi_{xy}+\frac{2}{U}(\Phi_x\Phi_{xt}+\Phi_y\Phi_{yt})\right] \tag{10}$$

For purposes of a second-order solution (and to higher order a potential does not exist), the triple products on the right-hand side can be disregarded. Thus, the perturbation equation becomes finally

$$\Phi_{yy}-\beta^2\Phi_{xx}-2\frac{M^2}{U}\Phi_{xt}-\frac{M^2}{U^2}\Phi_{tt}=M^2\left[(\gamma-1)\left(\Phi_x+\frac{\Phi_t}{U}\right)(\Phi_{xx}+\Phi_{yy})+2\Phi_x\Phi_{xx}+2\Phi_y\Phi_{xy}+\frac{2}{U}(\Phi_x\Phi_{xt}+\Phi_y\Phi_{yt})\right] \tag{11}$$

where $\beta^2=M^2-1$.

PRESSURE RELATION

Dividing the Bernoulli equation (eq. (5)) by $a^2/(\gamma-1)$ gives

$$\frac{a^2}{a_0^2}=1+\frac{\gamma-1}{2}M^2\left(1-\frac{u^2+v^2+2\Omega t}{U^2}\right) \tag{12}$$

[2] In the previous literature, both American and British, this is often designated as the angle of attack α, since the two are equivalent for a fixed pivot. However, it is θ but not α in the general case of a moving pivot, according to the usual notation of stability analysis.

The flow is isentropic to second order, so that

$$\frac{p}{p_0}=\left(\frac{a^2}{a_0^2}\right)^{\frac{\gamma}{\gamma-1}},\quad \frac{p_0}{\frac{1}{2}\rho_0 U^2}=\frac{2}{\gamma M^2}$$

and it follows that the pressure coefficient at any point is given by

$$C_p\equiv\frac{p-p_0}{\frac{1}{2}\rho_0 U^2}=\frac{2}{\gamma M^2}\left\{\left[1+\frac{\gamma-1}{2}M^2\left(1-\frac{u^2+v^2+2\Omega_t}{U^2}\right)\right]^{\frac{\gamma}{\gamma-1}}-1\right\} \tag{13}$$

Substituting the velocity components of equation (9b), expanding in series, and retaining only squares and products of perturbation quantities gives finally

$$C_p=-2\Phi_x-2\frac{\Phi_t}{U}+\beta^2\Phi_x^2-\Phi_y^2+2M^2\Phi_x\frac{\Phi_t}{U}+M^2\left(\frac{\Phi_t}{U}\right)^2 \tag{14}$$

Here the second-order solution is required only for evaluating the first two terms; the others are given correct to second order by linear theory.

SMOOTHING OF PROBLEM

From the leading edge and from any subsequent corner of the airfoil spring shock waves or Prandtl-Meyer expansion fans that oscillate as the airfoil oscillates. These introduce serious complications into the second-order analysis. However, the complications can all be circumvented by solving a "smoothed" problem in place of the actual problem. The solution can thereupon be applied to the actual problem, for which it yields the correct result everywhere except near the shock waves and Prandtl-Meyer fans.

The nature of the difficulties can be understood by considering first the special problem of steady flow past a single wedge (fig. 2).

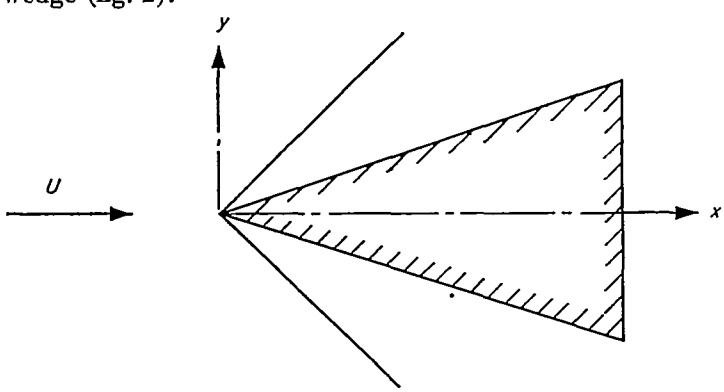


FIGURE 2.—Steady flow past a wedge.

The presence of the bow shock wave means that the analysis must be undesirably complicated by including the Rankine-Hugoniot relations (in a simplified form). A second complication arises in the differential equation which, for steady flow, becomes

$$\Phi_{yy}-\beta^2\Phi_{xx}=M^2[(\gamma+1)\Phi_x\Phi_{xx}+(\gamma-1)\Phi_x\Phi_{yy}+2\Phi_y\Phi_{xy}] \tag{15}$$

In the iteration procedure to be employed, the nonlinear right-hand side is evaluated in terms of the first-order solution, and the resulting nonhomogeneous wave equation solved for the second-order potential. However, for the wedge the right-hand side vanishes (to any order), which would imply incorrectly that the second-order solution does not involve the adiabatic exponent γ. More precisely, the right-hand side vanishes everywhere except along the Mach lines springing from the apex, where it has the singular behavior of the Dirac delta function, and only by taking account of these troublesome singularities could the correct solution be found.

Both these complications are avoided by the simple device of solving the problem of flow past a smooth cusp-nosed airfoil of arbitrary shape and then applying the final solution to the wedge. It may be imagined that the wedge has been smoothed by adding a cusped extension to its nose, as indicated in figure 3. It is clear that this artifice removes

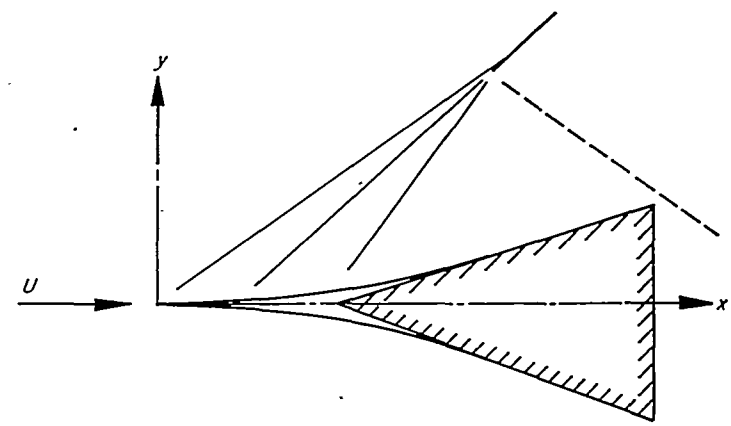


FIGURE 3.—Steady flow past a smoothed wedge.

the troublesome singularities from the right-hand side of the differential equation. Likewise, it eliminates the need for the shock-wave relations because, as indicated in figure 3, with sufficient smoothing, shock waves will form only at such great distance that their effects cannot reach the airfoil surface. Although shock waves are thus apparently excluded, the correct second-order result for the wedge is nevertheless recovered from the solution by imagining the extension to shrink in size and disappear. The reason is that to second order a shock wave is equivalent to the limit of a rapid continuous isentropic compression. This limiting procedure, which is equivalent simply to applying the solution for an arbitrary smooth shape to one that is not smooth, yields the proper result except in the vicinity of the shock wave (see ref. 5). For an airfoil of general shape, similar broad smoothing must be imagined at any concave corner; whereas at convex corners (since no shock waves form) the slightest rounding is enough. This smoothing technique was applied in reference 4 (p. 399) to steady first-order flow past bodies of revolution, and in reference 5 to steady second-order plane flow.

We turn now to the question of generalizing this smoothing scheme to an oscillating airfoil. Modification is necessary only at the leading edge. Consider first the special case of rotation about the leading edge. Then it is enough to conceive of an extension which is flexible, so that its cusped tip can be maintained fixed and directed always into the free

stream while the airfoil oscillates, as indicated in figure 4. (The exact motion of the flexible tip is immaterial, provided the surface is sufficiently smooth and its slope remains small.) After the solution has been found, the flexible extension is again imagined to shrink away, and the correct result is recovered for the actual airfoil oscillating about its nose.

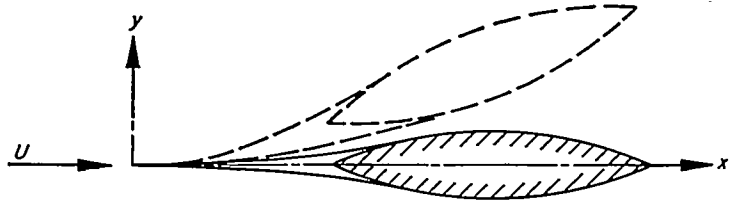


FIGURE 4.—Smoothing for airfoil oscillating about leading edge.

Finally, consider rotation about an arbitrary point. The flexible extension must now oscillate in such a way that its tip is always directed into the relative wind. Hence, as indicated in figure 5, the tip must lie parallel with the free stream at the top (and bottom) of each stroke but incline in the direction of motion for intermediate positions.

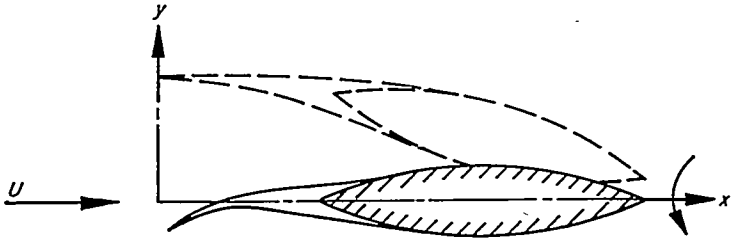


FIGURE 5.—Smoothing for airfoil oscillating about arbitrary point.

We are accordingly led to consider the motion of an arbitrary flexible oscillating surface described by

$$y=\epsilon f(x)-\theta_0 e^{i\omega t}g(x) \tag{16a}$$

where for the smoothed problem the functions $f(x)$ and $g(x)$ have continuous first derivatives. The smoothed problem will ultimately be replaced by the actual problem. According to comparison with equation (3), this means that the function $g(x)$ will eventually be identified with $(x-b)$. The requirement that the leading edge of the smoothed shape be always parallel to the relative wind may be written as

$$\left.\begin{aligned} f'(0)&=0\\ g'(0)&=i\omega b/U\end{aligned}\right\} \tag{16b}$$

as is clear from equation (19) of the next section. (These last conditions, as well as the requirement that $f'(x)$ be continuous, must be relaxed in recovering the solution of the actual problem.)

BOUNDARY CONDITIONS

The boundary condition at the surface of the airfoil is that the normal component of velocity is zero. For any surface described by $S(x,y,t)=0$ moving through a velocity field $\vec{V}$, this condition means that the substantial derivative of S (i. e., its time rate of change for an observer moving with the fluid) vanishes at the surface (see ref. 6), so that

$$S_t+\vec{V}\cdot\text{grad } S=0 \quad \text{at } S=0 \tag{17}$$

With velocity components given in terms of Φ by equation (9b), and for the smoothed surface described by equation (16a), this tangency condition becomes

$$\Phi_y=(1+\Phi_x)(\epsilon f'-\theta_0 e^{i\omega t}g')-\frac{i\omega}{U}\theta_0 e^{i\omega t}g \quad at \quad y=\epsilon f-\theta_0 e^{i\omega t}g \tag{18}$$

where $f\equiv f(x)$, etc. It is convenient to refer this condition to the axis $y=0$ by expanding in Taylor series. Keeping only terms of second order gives

$$\Phi_y=(1+\Phi_x)(\epsilon f'-\theta_0 e^{i\omega t}g')-\frac{i\omega}{U}\theta_0 e^{i\omega t}g-(\epsilon f-\theta_0 e^{i\omega t}g)\Phi_{yy} \text{ at } y=0 \tag{19}$$

(Here Φ_x and Φ_{yy} on the right-hand side can be evaluated from linearized theory.)

The upstream boundary condition requires that in the actual problem, the Rankin-Hugoniot relations (or at least a simplified second-order form thereof) be satisfied across an oscilliating bow shock wave whose position must be determined. However, shock waves have been eliminated from the smoothed problem, so that it is only necessary to require that the perturbation potential Φ vanish along the oscillating characteristic line (Mach line) springing from the leading edge. This insures that all disturbances produced by the airfoil are swept downstream. An equivalent and still simpler requirement is that Φ and its streamwise derivative vanish on, say, the plane $x=0$:

$$\Phi=\Phi_x=0 \quad \text{at } x=0 \tag{20}$$

TRANSFORMATION OF PERTURBATION EQUATION

It is convenient to separate the time-dependent part of the problem from the mean steady flow at zero angle of attack (for which the second-order solution is known). Furthermore, for harmonic oscillations the number of independent variables is then reduced to two by separating an exponential time factor. Finally, the linear portion of the time-dependent equation is reduced to normal form by a transformation of dependent variable. These three transformations amount to setting

$$\Phi(x,y,t)=\phi(x,y)+\theta_0 e^{i(\omega t-\kappa x)}\Psi(x,y) \tag{21a}$$

where

$$\kappa=\frac{M^2}{\beta^2}\frac{\omega}{U} \tag{21b}$$

Here ϕ corresponds to the mean steady flow, and the term in Ψ represents the additional flow associated with the oscillation through small angle of attack.

Introducing this transformation into the perturbation equation (eq. (11)) gives for the potential ϕ of the mean steady flow

$$\phi_{yy}-\beta^2\phi_{xx}=M^2[(\gamma-1)\phi_x\Delta\phi+(\phi_x{}^2+\phi_y{}^2)_x] \tag{22}$$

where Δ is the Laplacian operation $\partial^2/\partial x^2+\partial^2/\partial y^2$, and for the time-dependent part Ψ

$$\Psi_{yy}-\beta^2\Psi_{xx}-\left(\frac{\beta\kappa}{M}\right)^2\Psi=M^2[(\gamma-1)(\phi_x\Delta\Psi+\Psi_x\Delta\phi)+2(\phi_x\Psi_x+\phi_y\Psi_y)_x]-i\kappa M^2\left[(\gamma-1)\left(2\phi_x\Psi_x+\frac{1}{M^2}\Psi\Delta\phi\right)+2\left(\frac{M^2+1}{M^2}\phi_x\Psi_x+\phi_{xx}\Psi+\frac{1}{M^2}\phi_y\Psi_y\right)\right]-\kappa^2[2+(\gamma-1)M^2]\phi_x\Psi \quad (23)$$

The tangency condition of equation (19) likewise separates into the two conditions

$$\phi_y=\epsilon(1+\phi_x)f'-\epsilon\phi_{yy}f \quad \text{at } y=0 \quad (24)$$

$$\Psi_y=-e^{i\kappa x}(1+\phi_x)g'-i\kappa\frac{\beta^2}{M^2}e^{i\kappa x}g+\epsilon(\Psi_x-i\kappa\Psi)f'-\epsilon\Psi_{yy}f+e^{i\kappa x}\phi_{yy}g \quad \text{at } y=0 \quad (25)$$

For the actual problem the second of these becomes, identifying $g(x)$ with $(x-b)$,

$$\Psi_y=-e^{i\kappa x}(1+\phi_x)-i\kappa\frac{\beta^2}{M^2}e^{i\kappa x}(x-b)+\epsilon(\Psi_x-i\kappa\Psi)f'-\epsilon\Psi_{yy}f+e^{i\kappa x}\phi_{yy}(x-b) \quad \text{at } y=0 \quad (26)$$

For pressures at the surface of the airfoil, the relation of equation (14) can be expressed in terms of values at $y=0$ by Taylor series expansion, with the result that to second order in thickness and first order in angle of attack

$$C_{p_u}=(-2\phi_x-2\epsilon\phi_{xy}f+\beta^2\phi_x^2-\phi_y^2)+2\theta_0e^{i(\omega t-\kappa x)}\left[\frac{i\kappa}{M^2}\Psi-\Psi_x+\epsilon\left(\frac{i\kappa}{M^2}\Psi_y-\Psi_{xy}\right)f+\beta^2\phi_x\Psi_x-\phi_y\Psi_y+e^{i\kappa x}\phi_{xy}g\right] \quad (27)$$

where all terms are to be evaluated at $y=0$.

SOLUTION BY ITERATION

Although the equation for ϕ is nonlinear, that for Ψ is linear, but with nonconstant coefficients depending upon ϕ. This corresponds to the physical concept that because of the restriction to linear terms in angle of pitch the oscillatory part of the flow is an acoustic field with, however, the speed of sound varying from point to point in accordance with the mean steady flow.

The well-known linearized or first-order theory results from disregarding the right-hand sides of equations (22) and (23). Thus, with the first-order potentials denoted by the lower case letters φ and ψ, the perturbation equations become

$$\varphi_{yy}-\beta^2\varphi_{xx}=0 \quad (28)$$

$$\psi_{yy}-\beta^2\psi_{xx}-\left(\frac{\beta\kappa}{M}\right)^2\psi=0 \quad (29)$$

The second-order solution is obtained by iterating upon the first-order results. Using the linear equations to simplify the right-hand sides gives for the second-order iteration equations

$$\phi_{yy}-\beta^2\phi_{xx}=2M^2[\beta^2(N-1)\varphi_x^2+\varphi_y^2]_x \quad (30)$$

and

$$\Psi_{yy}-\beta^2\Psi_{xx}-\left(\frac{\beta\kappa}{M}\right)^2\Psi=2M^2[\beta^2(N-1)\varphi_x\psi_x+\varphi_y\psi_y]_x-2i\kappa[\beta^2(2N-1)\varphi_x\psi_x+\beta^2N\varphi_{xx}\psi+\varphi_y\psi_y]-2N\left(\frac{\beta\kappa}{M}\right)^2\varphi_x\psi \quad (31)$$

where

$$N=\frac{\gamma+1}{2}\frac{M^2}{\beta^2} \quad (32)$$

(Here, following the usual subscript notation for derivatives, $[\varphi_y^2]_x$ means $\partial\varphi_y^2/\partial x$, etc.) The second-order solution for Φ, which leads to Busemann's well-known result at the airfoil surface (ref. 1), was given in reference 5. It is therefore necessary to consider only the second-order problem for Ψ. Details of the interation procedure and discussion of its limitations are given for the steady flow in reference 5 and apply also to the present problem.

PARTIAL PARTICULAR INTEGRAL

The solution of the differential equation for plane or axially symmetric steady flow in reference 5 was simplified by discovery of a particular integral of the iteration equation in terms of the first-order solution. It was also shown there that for steady three-dimensional flow a particular integral can be found to account for all terms in the iteration equation except those involving the adiadatic exponent γ in the form of N. Likewise, here, a partial particular integral that accounts for all terms on the right-hand side of equation (31) except those involving N is given by

$$\Psi_*=M^2(\varphi\psi)_x-i\kappa\varphi\psi \quad (33)$$

The complete solution is this partial particular integral plus a solution of the reduced equation whose right-hand side contains only the terms still unaccounted for:

$$\Psi_{yy}-\beta^2\Psi_{xx}-\left(\frac{\beta\kappa}{M}\right)^2\Psi=2\beta^2N\left\{M^2(\varphi_x\psi_x)_x-i\kappa[(\varphi_x\psi)_x+\varphi_x\psi_x]-\frac{\kappa^2}{M^2}\varphi_x\psi\right\} \quad (34)$$

FIRST-ORDER SOLUTION

The first-order solution for φ is known from Ackeret's theory to be

$$\varphi=-\frac{\epsilon}{\beta}f(x-\beta y) \quad (35)$$

It is to be understood here and in all similar expressions to follow that this is the potential only for $x\geq\beta y$, and that φ vanishes identically ahead of the bow Mach wave (where $x\leq\beta y$).

The first-order equation for ψ (eq. (29)) is most readily solved by applying the Laplace transformation with respect to x. We denote the Laplace transform of a function either by a bar, or by the symbol $\mathcal{L}$, whichever is more convenient (and the inverse transform by $\mathcal{L}^{-1}$), so that, for example

$$\left.\begin{matrix}\bar{w}(s)\\ \mathcal{L}\{w(x)\}\end{matrix}\right\}=\int_0^\infty e^{-sx}w(x)dx \quad (36)$$

Applying this transformation to equation (29), using the fact that ψ and ψ_x vanish at $x=0$, gives

$$\bar{\psi}_{yy}-\beta^2\left(s^2+\frac{\kappa^2}{M^2}\right)\bar{\psi}=0 \tag{37}$$

The solution of this equation that represents waves moving downstream is

$$\bar{\psi}=C(s)\,e^{-\beta y\sqrt{s^2+\kappa^2/M^2}} \tag{38}$$

The coefficient $C(s)$ is determined by the first-order form of the tangency condition (eq. (25)), which transforms to

$$\bar{\psi}_y=-\bar{w}(s) \quad \text{at } y=0 \tag{39a}$$

where

$$w(x)=e^{i\kappa x}\left[g'(x)+i\kappa\frac{\beta^2}{M^2}g(x)\right] \tag{39b}$$

is the downwash velocity at $y=0$. Consequently,

$$\bar{\psi}=\frac{1}{\beta}\bar{w}(s)\frac{e^{-\beta y\sqrt{s^2+\kappa^2/M^2}}}{\sqrt{s^2+\kappa^2/M^2}} \tag{40}$$

The inverse transformation is readily carried out using the standard tables (e. g., ref. 7) together with the convolution theorem, which gives as the solution of the smoothed problem

$$\psi(x,y)=\frac{1}{\beta}\int_0^{x-\beta y}J_0\left(\frac{\kappa}{M}\sqrt{\xi^2-\beta^2y^2}\right)w(x-\xi)\,d\xi \tag{41}$$

The solution for the actual problem is now obtained by setting $g(x)=x-b$, which gives finally [3]

$$\psi(x,y)=\frac{1}{\beta}\int_0^{x-\beta y}J_0\left(\frac{\kappa}{M}\sqrt{\xi^2-\beta^2y^2}\right)e^{i\kappa(x-\xi)}\left[1+i\kappa\frac{\beta^2}{M^2}(x-\xi-b)\right]d\xi \tag{42}$$

in agreement with the known result of linearized theory (see, e. g., ref. 8).

SECOND-ORDER SOLUTION FOR LOW FREQUENCIES

Because the first-order solution for arbitrary frequencies is rather complicated, use is sometimes found for an expansion in powers of frequency, which involves only elementary functions. The corresponding second-order solution will now be carried out in detail, including linear terms in frequency. This result will serve, for example, to evaluate the effects of thickness upon one-degree-of-freedom torsional instability, which is primarily a low-frequency effect. Thereafter, the result of extending the solution to include third powers of frequency will simply be stated.

POTENTIAL INCLUDING LINEAR TERMS IN FREQUENCY

Expanding the first-order solution of equation (41) in powers of the frequency parameter κ and retaining only linear terms gives

$$\psi(x,y)=\frac{1}{\beta}\int_0^z w(\xi)d\xi+0(\kappa^2)=\psi(z)+\ldots \tag{43}$$

where $z=x-\beta y$. To this order the partial particular integral of equation (33) is a solution of the homogeneous equation (eq. (29)), and can therefore be disregarded. Substituting the first-order solutions into the right-hand side of equation (34) and applying the Laplace transformation gives, to order κ,

$$\bar{\Psi}_{yy}-\beta^2s^2\bar{\Psi}=-2\epsilon Ne^{-\beta sy}[(M^2s-i\kappa)\mathcal{L}\{f'(x)\psi'(x)\}-i\kappa s\mathcal{L}\{f'(x)\psi(x)\}] \tag{44}$$

It is readily found that a particular integral of this equation is given by an appropriate multiple of $ye^{-\beta sy}$. Then, adding a complementary function representing downgoing waves gives

$$\bar{\Psi}=C(s)e^{-\beta sy}+\frac{\epsilon N}{\beta}ye^{-\beta sy}\left[\left(M^2-\frac{i\kappa}{s}\right)\mathcal{L}\{f'\psi'\}-i\kappa\mathcal{L}\{f'\psi\}\right] \tag{45}$$

where the coefficient $C(s)$ is to be evaluated from the tangency condition. Inverting the Laplace transformation shows that for the actual problem, in which $g(x)=x-b$, the solution has the form

$$\Psi=F(z)+\frac{\epsilon N}{\beta}y[M^2f'+i\kappa(2\beta^2zf'-\beta^2bf'-f)] \tag{46a}$$

where $f\equiv f(z)$. The arbitrary function $F(z)$ is determined from the tangency condition of equation (26) to be

$$F(z)=\frac{z}{\beta}+\frac{i\kappa}{\beta}z\left(\frac{2M^2-1}{2M^2}z-\frac{\beta^2}{M^2}b\right)+\epsilon\left[\frac{M^2N-M^2-1}{\beta^2}f+(x-b)f'\right]+i\kappa\epsilon\left\{\left[2-(N-1)\frac{2M^2-1}{\beta^2}\right]\int_0^z f(\xi)d\xi+\left[2(N-1)-\frac{2M^2-1}{M^2\beta^2}\right]zf+z(z-b)f'+\left[\frac{1}{M^2}-(N-1)\right]bf\right\} \tag{46b}$$

SURFACE PRESSURE COEFFICIENT

The pressure coefficient at the upper surface of the actual airfoil is found from equation (27) to be

$$C_{p_u}=C_{p_0}+2\theta_0e^{i\omega t}\left\{-\frac{1}{\beta}+i\kappa\frac{\beta}{M^2}\left(b+\frac{2-M^2}{\beta^2}x\right)-\frac{M^2N-2}{\beta^2}\epsilon f'+i\kappa\frac{\epsilon}{M^2}\left[2M^2\frac{N-1}{\beta^2}f+\frac{(2-M^2)(M^2N-1)}{\beta^2}xf'+(M^2N-2)bf'\right]\right\}+0(\kappa^2,\epsilon^2\theta,\theta^2) \tag{47}$$

where $f\equiv f(x)$. Here C_{p_0} is the value for the mean steady flow (at zero angle of pitch), which is given by Busemann's second-order theory. A more useful form of the result is obtained by extracting the real part and expressing the result in terms of the instantaneous angle of pitch $\theta(t)$ and its time rate of change $\dot{\theta}(t)$. Furthermore, the parameter ϵ has served its purpose of distinguishing terms of different orders and can be eliminated (according to eq. (1)). Thus, on the upper surface of an arbitrary airfoil that is described at zero angle of pitch by $y=Y(x)$, is pivoted about a point a distance b downstream of its leading edge, and performs slow angular oscillations described by $\theta(t)$, the pressure

[3] The smoothing was, of course, unnecessary in this first-order problem.

coefficient is, to second order in thickness and first order in angle of pitch,

$$C_{p_u}=C_{p_0}-\frac{2}{\beta}\theta+\frac{2}{\beta}\left(\frac{2-M^2}{\beta^2}x+b\right)\frac{\dot\theta}{U}-2\frac{M^2N-2}{\beta^2}Y'\theta+$$
$$2\left[\frac{2M^2(N-1)}{\beta^4}Y+\frac{(2-M^2)(M^2N-1)}{\beta^4}xY'+\right.$$
$$\left.\frac{M^2N-2}{\beta^2}bY'\right]\frac{\dot\theta}{U}+0(\ddot\theta,\epsilon^2\theta,\theta^2) \tag{48a}$$

Here the value for the mean steady flow is (ref. 1)

$$C_{p_0}=\frac{2}{\beta}Y'+\frac{M^2N-2}{\beta^2}Y'^2 \tag{48b}$$

(A preliminary report of this result was given in ref. 9.) In this form, the result is not restricted to sinusoidal motion but applies to any oscillation that is sufficiently smooth and slow that the pressures depend significantly only upon the instantaneous angle of pitch and angular velocity.

The pressure on the lower surface of the airfoil is obtained from these equations by reversing the sign of θ, and taking $Y(x)$ to be the ordinate of the lower surface, measured positive downward.

The result for plunging motions can be extracted by letting θ tend to zero and b tend to infinity in such a way that their product remains finite, say

$$b\theta(t)=-h(t)=-h_0e^{i\omega t} \tag{49}$$

In the limit, the airfoil simply translates vertically according to $y=-h(t)$. The pressure coefficient on the upper surface is [4]

$$C_{p_u}=C_{p_0}-\frac{2}{\beta}\frac{\dot h}{U}-2\frac{M^2N-2}{\beta^2}Y'\frac{\dot h}{U} \tag{50}$$

CHECKS ON THE RESULT

The solution can be tested in several special cases for which the result can be derived from other considerations.

Of the five terms in equation (48a), the first is known from Busemann's steady second-order solution, and the second and third from linearized unsteady theory. The fourth is obtained by using the instantaneous airfoil slope $(Y'-\theta)$ instead of the mean steady slope Y' in Busemann's formula and retaining only linear terms in θ. Therefore, only the last term, which is the essentially new result of the present analysis, requires verification.

Just at the nose of an oscillating airfoil, the pressure can be determined exactly if the bow shock wave is attached. The transition through the moving bow shock is instantaneous, and so depends only upon the relative velocity at that instant (see ref. 4, p. 297). Hence the pressure just at the nose is instantaneously the same as on a wedge of the same vertex angle in steady flow with the same relative velocity. In the present problem, the relative velocity is compounded of the horizontal velocity U of the free stream

[4] Note that, as it should be, this is just the result of using Busemann's formula for steady flow (eq. (48b)), with the local slope decreased by the instantaneous apparent angle $\dot h/U$; see the discussion in the following section of conditions at the nose.

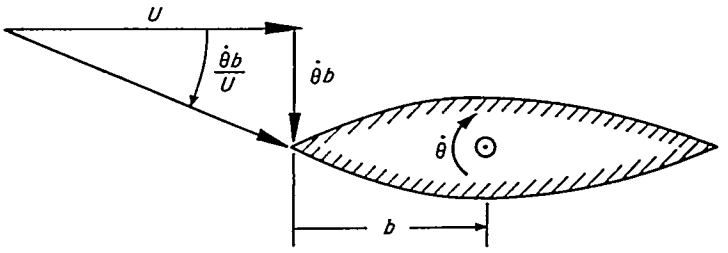


FIGURE 6.—Velocity relative to leading edge of oscillating airfoil.

plus the instantaneous vertical velocity of the leading edge, which is given by $\dot\theta b$ (see fig. 6). The effect of the vertical component upon the equivalent free-stream velocity and Mach number is of second order in angle of pitch, but the equivalent vertex angle of the airfoil is increased by the apparent downwash angle $\dot\theta b/U$. Replacing Y' by $Y'+\dot\theta b/U$ in Busemann's formula (eq. (48b)) gives, to first order in angle of pitch

$$C_{p_u}=C_{p_0}+\frac{2}{\beta}b\frac{\dot\theta}{U}+2\frac{M^2N-2}{\beta^2}bY'\frac{\dot\theta}{U} \tag{51}$$

which checks the part proportional to b of the last term in equation (48a).

The remainder of the term in question can be checked for a single-wedge airfoil oscillating about its vertex (fig. 7). It

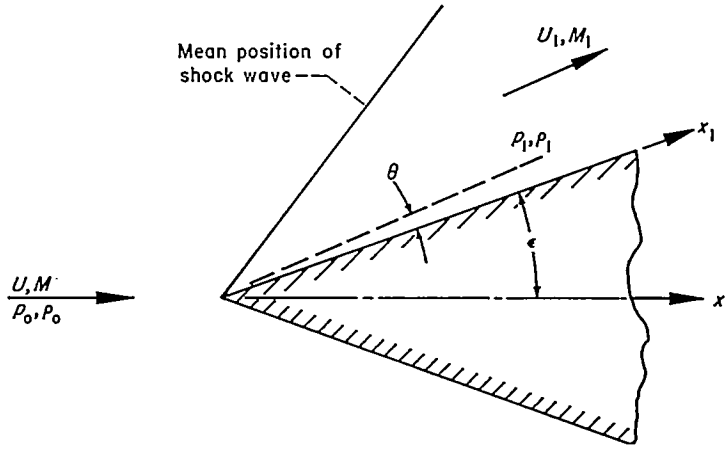


FIGURE 7.—Wedge oscillating about its vertex.

can be shown using the results of reference 3 that in this case disturbances reflected from the shock wave are of third order in the wedge angle (although for other pivot positions they are of second order). Therefore, a solution correct to second order in thickness and first order in angle of pitch can be found by applying linearized theory to the mean steady flow behind the shock wave.[5] For slow oscillations, the first three terms of equation (48a) give

$$\frac{p-p_1}{\frac{1}{2}\rho_1U_1^2}=-\frac{2}{\beta_1}\theta+\frac{2(2-M_1^2)}{\beta_1^3}x_1\frac{\dot\theta}{U_1} \tag{52}$$

[5] This concept was suggested to the author by W. P. Jones of the National Physical Laboratory, England.

where subscript 1 denotes values in the mean steady flow behind the shock wave. From linearized theory

$$\left.\begin{aligned} M_1&=M\ [1-\beta(N-1)\epsilon] \\ \beta_1&=\beta\ \left[1-\frac{M^2}{\beta}(N-1)\epsilon\right] \\ U_1&=U\left(1-\frac{\epsilon}{\beta}\right) \\ p_1&=p_0\left(1+\gamma\frac{M^2}{\beta}\epsilon\right) \\ \rho_1&=\rho_0\left(1+\frac{M^2}{\beta}\epsilon\right) \end{aligned}\right\} \tag{53}$$

where ϵ is the semivertex angle. Hence, referring the surface pressure coefficient to free-stream quantities (and noting that $x_1=x$ to second order) gives

$$C_{p_u}=C_{p_0}-\frac{2}{\beta}\theta+2\frac{2-M^2}{\beta^3}x\frac{\dot\theta}{U}-2\frac{M^2N-2}{\beta^2}\epsilon\theta+ 2\frac{2M^2(N-1)+(2-M^2)(M^2N-1)}{\beta^4}\epsilon x\frac{\dot\theta}{U} \tag{54}$$

which checks the first two parts of the last term in equation (48a) when $Y(x)=\epsilon x$.

Recently, Lighthill has given a further check for the case of Mach numbers so high that $1/M^2$ is negligible compared with unity (ref. 10).

LIFT AND MOMENT COEFFICIENTS FOR SYMMETRICAL AIRFOILS

The coefficients of lift and pitching moment (about the pivot) are given in terms of the pressure coefficients on the upper and lower surfaces by

$$c_l=\frac{1}{c}\int_0^c(C_{p_l}-C_{p_u})\,dx \tag{55a}$$

$$c_m=\frac{1}{c^2}\int_0^c(b-x)(C_{p_l}-C_{p_u})\,dx \tag{55b}$$

For simplicity, consider only airfoils symmetric about the chord line. In this case, the pressure difference is given by

$$C_{p_l}-C_{p_u}=\frac{4}{\beta}\theta-\frac{4}{\beta}\left(\frac{2-M^2}{\beta^2}x+b\right)\frac{\dot\theta}{U}+4\frac{M^2N-2}{\beta^2}Y'\theta- 4\left[\frac{2M^2(N-1)}{\beta^4}Y+\frac{(2-M^2)(M^2N-1)}{\beta^4}xY'+ \frac{M^2N-2}{\beta^3}bY'\right]\frac{\dot\theta}{U} \tag{56}$$

If the airfoil has a sharp trailing edge, substituting into equations (55) and integrating by parts gives

$$c_l=\frac{4}{\beta}\theta-\frac{4}{\beta}\left[\left(\frac{2-M^2}{2\beta^2}+\frac{b}{c}\right)+\frac{M^4N-3M^2+2}{\beta^3}\frac{1}{c^2}\int_0^c Y\,dx\right]\frac{c\dot\theta}{U} \tag{57}$$

and

$$c_m=\frac{4}{\beta}\left[\left(\frac{b}{c}-\frac{1}{2}\right)+\frac{M^2N-2}{\beta}\frac{1}{c^2}\int_0^c Y\,dx\right]\theta+ \frac{4}{\beta}\left[\left(\frac{2-M^2}{3\beta^2}+\frac{2M^2-3}{2\beta^2}\frac{b}{c}-\frac{b^2}{c^2}\right)-\frac{M^2(N-1)}{\beta^3}\frac{b}{c}\times \frac{1}{c^2}\int_0^c Y\,dx+2\frac{M^2N-2}{\beta}\frac{1}{c^3}\int_0^c(x-b)Y\,dx\right]\frac{c\dot\theta}{U} \tag{58}$$

Inside each square bracket the first term is the result of linearized theory, and the remainder represents the second-order effect of thickness. Thickness effects are seen to appear in the form of the area of the airfoil profile and its first moment about the vertical line through the pivot.

If the airfoil has a blunt trailing edge of semithickness $Y(c)$, the following additional second-order terms must be added to the above expressions:

to c_l:

$$\frac{4}{\beta}\frac{M^2N-2}{\beta}\frac{Y(c)}{c}\theta-\frac{4}{\beta}\left[\frac{M^2N-2}{\beta}\frac{b}{c}+\frac{(2-M^2)(M^2N-1)}{\beta^3}\right]\frac{Y(c)}{c}\frac{c\dot\theta}{U} \tag{59}$$

to c_m:

$$-\frac{4}{\beta}\frac{M^2N-2}{\beta}\left(1-\frac{b}{c}\right)\frac{Y(c)}{c}\theta+ \frac{4}{\beta}\left[\frac{M^2(N-1)}{\beta^3}-\frac{M^2N-2}{\beta}\left(1-\frac{b}{c}\right)\right]\left(1-\frac{b}{c}\right)\frac{Y(c)}{c}\frac{c\dot\theta}{U} \tag{60}$$

EXAMPLES: BICONVEX AND DOUBLE-WEDGE AIRFOILS

To second order, a biconvex airfoil of thickness ratio τ is given by the parabolas

$$y=\pm Y(x),\qquad Y(x)=2\tau\frac{x}{c}(c-x) \tag{61}$$

The expressions for lift and moment become

$$c_l=\frac{4}{\beta}\theta-\frac{4}{\beta}\left[\left(\frac{2-M^2}{2\beta^2}+\frac{b}{c}\right)+\frac{M^4N-3M^2+2}{3\beta^3}\tau\right]\frac{c\dot\theta}{U} \tag{62}$$

$$c_m=\frac{4}{\beta}\left[\left(\frac{b}{c}-\frac{1}{2}\right)+\frac{M^2N-2}{3\beta}\tau\right]\theta+\frac{4}{\beta}\left[\left(\frac{2-M^2}{3\beta^2}+\frac{2M^2-3}{2\beta^2}\frac{b}{c}-\frac{b^2}{c^2}\right)+ \frac{M^2N-2}{3\beta}\left(1-2\frac{b}{c}\right)\tau-M^2\frac{N-1}{3\beta^3}\frac{b}{c}\tau\right]\frac{c\dot\theta}{U} \tag{63}$$

It happens that for a double-wedge airfoil (with maximum thickness at midchord), both the area and first moment are just three fourths of those for the biconvex airfoil. Consequently, the above results apply to double-wedge airfoils if $\tau/3$ is replaced throughout by $\tau/4$.

In the expression for pitching moment, the term proportional to θ represents an aerodynamic stiffness or restoring moment in phase with the angular displacement, while the term proportional to $\dot\theta$ corresponds to an aerodynamic damping moment in phase with the angular velocity. The effects of thickness upon aerodynamic restoring and damping moments are shown in figures 8 and 9 for a 6-percent-thick double-wedge or a 4½-percent-thick biconvex airfoil with

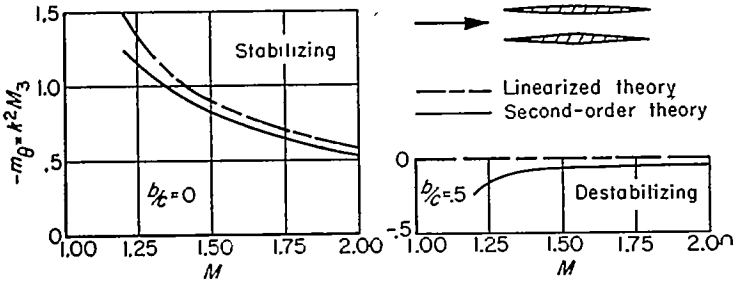


FIGURE 8.—Effect of thickness upon restoring-moment coefficient for 4½-percent-thick biconvex or 6-percent-thick double-wedge airfoil.

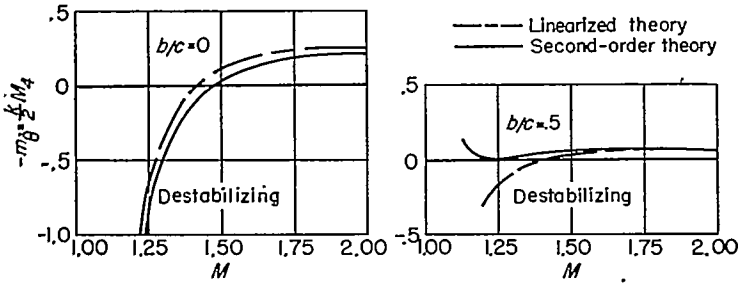


FIGURE 9.—Effect of thickness upon damping-moment coefficient for 4½-percent-thick biconvex or 6-percent-thick double-wedge airfoil.

two different pivot positions. The figures have been labeled with both the usual American notation (e. g., ref. 11) and the British notation (e. g., ref. 12) for flutter derivatives, defined for harmonic oscillations of arbitrary frequency by

$$\frac{c_m}{4\theta_0 e^{i\omega t}}=\begin{Bmatrix}-\frac{1}{2}k^2(M_3+iM_4) & \text{(American)}\\ -\frac{1}{2}(m_\theta+i\lambda m_{\dot\theta}) & \text{(British)}\end{Bmatrix} \tag{64}$$

Here k and λ are the reduced frequency in the American and British notations, respectively, related by

$$\lambda=2k=\frac{\omega c}{U} \tag{65}$$

For slow oscillations, the flutter coefficients are obtained from equation (63) according to

$$\left.\begin{aligned} m_\theta&=-k^2M_3=\frac{1}{2}\frac{\partial c_m}{\partial\theta}\\ m_{\dot\theta}&=-\frac{k}{2}M_4=\frac{U}{2c}\frac{\partial c_m}{\partial\dot\theta}\end{aligned}\right\} \tag{66}$$

(and, indeed, this was originally the definition of m_θ and $m_{\dot\theta}$, that given by equation (64) being a later extension to the case of rapid harmonic oscillations).

It should be noted that according to second-order theory the nonlinear effects of thickness are themselves linear in thickness. This means, for example, that doubling the airfoil thickness ratio would double the distance between the linearized and second-order curves of figures 8 and 9.

NEUTRAL DAMPING BOUNDARY

Linearized theory indicates the possibility of instability of pitching oscillations for low frequencies. For a range of Mach numbers below $\sqrt{5/2}=1.58$ and pivots ahead of two-thirds of the chord, the aerodynamic damping moment becomes negative, and so tends to destabilize. (Whether or not the motion is actually unstable depends, of course, upon the other dynamic parameters in the problem.) This zone of possible instability shrinks and eventually disappears as the frequency of oscillation increases. The present low-frequency solution is therefore adequate for determining how the region of instability is modified by nonlinear thickness effects.

Figure 10 shows the boundary of neutral aerodynamic damping for slow oscillations of a 4½-percent-thick biconvex or 6-percent-thick double-wedge airfoil. The aerodynamic damping is destabilizing for Mach numbers and pivot positions lying inside the loops. Within the region where linearized theory predicts a destabilizing moment, thickness is seen to exert a further destabilizing effect except for pivots near midchord. The second-order solution becomes unreliable when the bow shock wave detaches, at about $M=1.2$.

COMPARISON WITH PREVIOUS INVESTIGATIONS

Two previous investigators have sought a second-order solution for slowly oscillating airfoils in supersonic flow. Their results agree neither with each other nor with the present solution.

In 1947, W. P. Jones obtained an estimate of the thickness effect by assuming that the ratio of second-order to linearized pressure disturbances is the same for slow oscillations as that given by Busemann's formula for steady flow (ref. 13). That this assumption is not altogether correct is indicated by the fact that the results do not check those obtained for a wedge oscillating about its vertex by applying linearized theory to the mean steady flow behind the shock wave. However, the assumption is correct at the leading edge, and also (as noted by Lighthill in ref. 10) in the limit of high Mach number. It is seen in figure 10 that this estimate fails to give a useful prediction of the actual effects of thickness, except for pivots near midchord.

In 1951, Alexander Wylly attacked the problem by

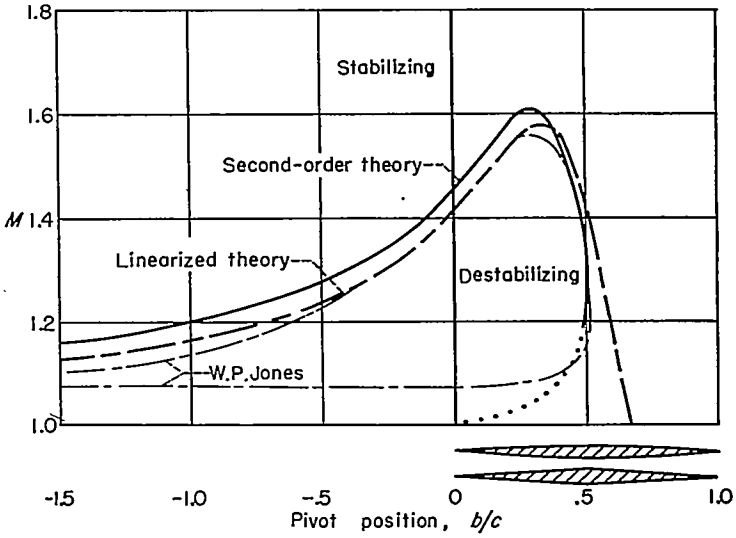


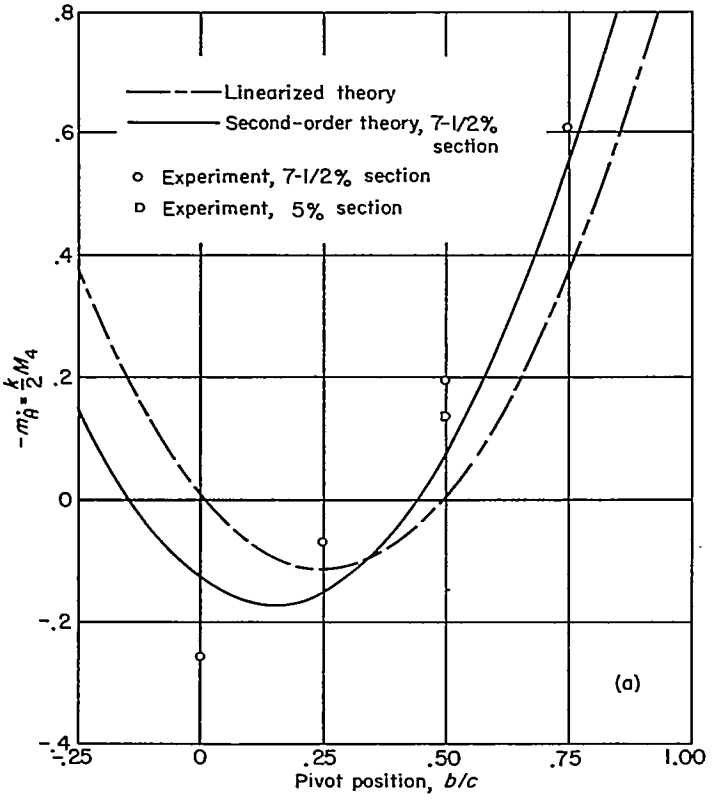
FIGURE 10.—Neutral damping boundary for 4½-percent-thick biconvex or 6-percent-thick double-wedge airfoil.

methods similar to those used here (ref. 14). Unfortunately, it appears that the smoothing was not carried out with sufficient care; as a consequence, the solution satisfies none of the three checks discussed previously. In contrast to the present results, the effect of thickness upon aerodynamic damping was predicted to be stabilizing and so great that for airfoils of the thicknesses shown in figure 10 the zone of possible instability would have disappeared altogether.

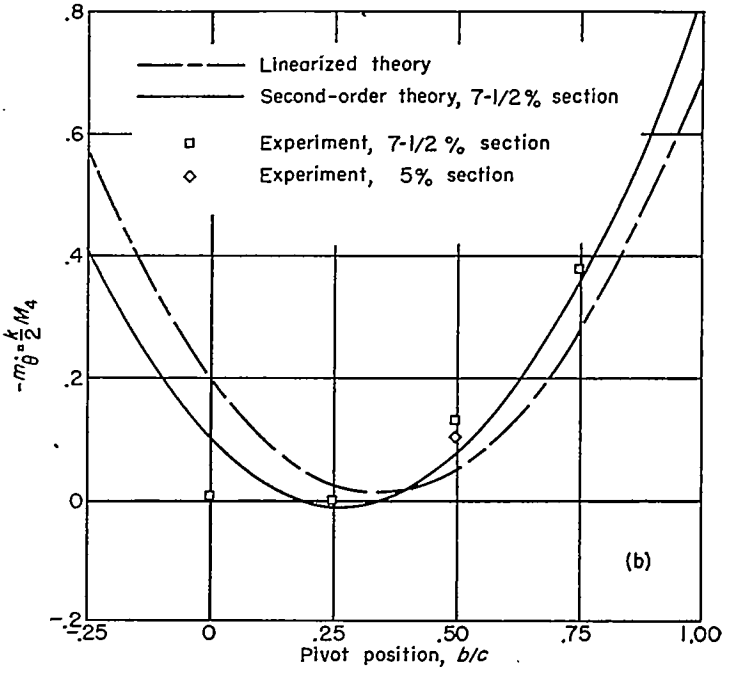
Concurrently with the original appearance of the present work, Martin and Gerber have published an independent investigation of the second-order effects of thickness on the stability derivatives for airfoils in constant vertical acceleration and constant pitching (refs. 15 and 16). Their results agree completely with those given above.

COMPARISON WITH EXPERIMENT

The first- and second-order theories are compared in figure 11 with the results of recent English experiments carried out at the National Physical Laboratory.[6] The aerodynamic damping at low frequencies was measured for biconvex airfoils 5 and 7.5 percent thick at Mach numbers of 1.42 and 1.61 with various pivot positions. For the points shown the amplitude of angular oscillation was 1.5°; increasing the amplitude to 3° was found to affect the results only at the lower Mach number. Including second-order thickness



(a) M=1.42



(b) M=1.61

FIGURE 11.—Comparison of experimental and theoretical damping-moment coefficients for biconvex airfoils.

[6] The author is indebted to W. P. Jones, J. B. Bratt, and W. E. A. Acum of the National Physical Laboratory for kindly making these data available in advance of publication. The work is to be published under the title "Measurements of Pitching Moment Derivatives for Two-Dimensional Models at Subsonic and Supersonic Speeds, and for a Rectangular Model of Aspect Ratio 4 at Subsonic Speeds," by Bratt, Raymer, and Townsend. (It may be noted that earlier experimental results from the National Physical Laboratory that were cited in a previous version of this paper (NACA TN 2332) are now believed by the experimenters to be unreliable.)

FIGURE 11.—Concluded.

effects in the theory is seen generally to improve the agreement with experiment. The lower Mach number is close to the limit of purely supersonic flow —M=1.38 for the 7.5-percent-thick section—at which the theory presumably breaks down.

EXTENSION TO CUBE OF FREQUENCY

The dependence of thickness effects upon frequency of oscillation can be estimated by extending the second-order solution to include higher powers of frequency. This has been carried out for an arbitrary airfoil by including second and third powers, which is enough to show an effect of frequency upon both aerodynamic stiffness and damping. The computation, though cumbersome, is a straightforward extension of the previous analysis, so that only the final result will be given here.

The expression for pressure coefficient on the upper surface, corresponding to equation (47), is found to be

$$\frac{C_{p_u}-C_{p_0}}{2\theta_0 e^{i\omega t}}=-\frac{1}{\beta}+\frac{i\kappa}{\beta}\left(\frac{2-M^2}{M^2}x+\frac{\beta^2}{M^2}b\right)+\frac{\kappa^2}{\beta}x\left(\frac{2+M^2}{4M^4}x+\frac{\beta^2}{M^4}b\right)-$$

$$\frac{i\kappa^3}{\beta}x^2\left(\frac{M^2+4}{12M^4}x+\frac{3\beta^2}{4M^4}b\right)-\epsilon\frac{M^2N-2}{\beta^2}f'+$$

$$i\kappa\epsilon\left[2\frac{N-1}{\beta^2}f+\frac{M^2N-2}{M^2}bf'+\right.$$

$$\left.\frac{(2-M^2)(M^2N-1)}{M^2\beta^2}xf'\right]+$$

$$\kappa^2\epsilon\left[\frac{3(3M^2-2)N-2(5M^2-3)}{2M^2\beta^2}\int f+\frac{4N-5}{2M^2}bf+\right.$$

$$\frac{(16-7M^2)N+4(2M^2-3)}{4M^2\beta^2}xf+\frac{N-1}{M^2}bxf'+$$

$$\left.\frac{(2+M^2)N-4}{4M^2\beta^2}x^2f'\right]+$$

$$ik^3\epsilon\left[\frac{(17M^4-10M^2-4)N-(5M^2-2)(4M^2-1)}{2M^4\beta^2}\int\xi f+\right.$$

$$\frac{6M^2-(5M^2-2)N}{2M^4}\,b\int f+$$

$$\frac{7M^2(2M^2-1)-(12M^4-3M^2-4)N}{2M^4\beta^2}\,x\int f+$$

$$\frac{2(M^2+1)-(M^2+8)N}{4M^4}\,bxf+$$

$$\frac{(3M^2+2)-(3M^2+4)N}{4M^4\beta^2}\,x^2f+$$

$$\frac{2(M^2+1)-3M^2N}{4M^4}\,bx^2f'+$$

$$\left.\frac{5M^2+2-M^2(4+M^2)N}{12M^4\beta^2}\,x^3f'\right] \tag{67}$$

where $f\equiv f(x)$ and $\int f\equiv\int_0^x f(\xi)\,d\xi$, etc. (The first four terms are the result of linearized theory.) This result meets the three tests discussed previously, and also checks the solution given later for a wedge with general pivot location at arbitrary frequency. The resulting expressions for lift and moment involve the airfoil thickness in the form of the area of the profile and its first three moments about the vertical line through the pivot.

EXAMPLE: BICONVEX AIRFOIL

These rather formidable results simplify considerably for specific airfoils. For example, for a biconvex airfoil of thickness ratio τ oscillating about midchord, the pitching-moment coefficient is given by

$$\frac{C_m}{4\theta_0 e^{i\omega t}}=i\lambda\,\frac{2-M^2}{12\beta^3}+\lambda^2\,\frac{M^2}{16\beta^5}-i\lambda^3\,\frac{M^2(6M^2-1)}{160\beta^7}+\tau\,\frac{M^2N-2}{3\beta^2}-$$

$$i\lambda\tau\,\frac{M^2(N-1)}{6\beta^4}+\lambda^2\tau M^2\,\frac{12(2-M^2)+(M^2-4)N}{240\beta^6}+$$

$$i\lambda^3\tau M^2\,\frac{2(48M^4-61\,M^2+1)-(37M^4+25M^2-16)N}{1440\beta^8} \tag{68}$$

The first three terms are the result of linearized theory.

The component of this moment that is out of phase with the angle of pitch gives a parabolic approximation for the variation of aerodynamic damping with frequency as shown by dashed lines in figure 12 for a 5-percent-thick biconvex airfoil. The accuracy of the parabolic approximation for linearized theory is indicated by comparing it with the exact result (solid line). In this example the linearized and second-order curves run almost parallel, which means that the nonlinear effects of thickness vary only slightly with frequency.

Recently, Jones and Skan have treated biconvex airfoils at arbitrary frequency by a numerical procedure (ref. 17). Their result is shown in figure 12. It fails to give the initially parabolic form of the curve that is implied by the fact that the second-order solution, like the first-order result, can be expanded in powers of the square of the frequency. Their

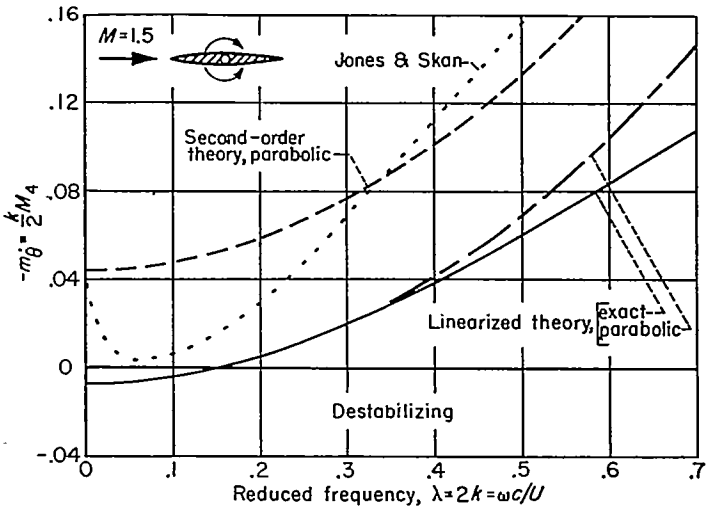


FIGURE 12.—Effect of frequency upon damping-moment coefficient for 5-percent-thick biconvex airfoil pivoted at midchord.

solution involves several doubtful assumptions, in particular, that the effect of the bow shock wave can be disregarded. It has already been remarked here that, actually, the bow shock has a second-order effect unless the pivot lies at the leading edge.

STABILITY DERIVATIVES

The oscillating motion of the airfoil has heretofore been described in terms of its angle of pitch θ and (negative) elevation h, as is customary in flutter analysis, and the flutter derivatives have been calculated. In stability analysis an alternative pair of coordinates is usually employed: the angle of attack α with respect to the relative wind and the rate of pitching $q=\dot\theta$. The motion is then no longer restricted to harmonic oscillations, so that there are actually an infinite number of stability derivatives. However, only the first three—c_{m_α}, c_{m_q}, $c_{m_{\dot\alpha}}$—for moment, and their counterparts for lift, are ordinarily considered significant.

Steady-flow theory gives c_{m_α}. It is shown in Appendix B (eq. (B12)) that the combination $(c_{m_q}+c_{m_{\dot\alpha}})$ is given by the solution for low frequencies, but that c_{m_q} and $c_{m_{\dot\alpha}}$ separately can be found only from the solution that includes the square of the frequency. For this purpose, only the solution for plunging is required, and this can be extracted from equation (67) as before by letting the pivot recede to infinity and the angular amplitude diminish according to equation (49), which gives

$$C_{p_u}-C_{p_l}=\frac{4h_0}{\beta^3U^2}\,\omega^2\left\{x+\frac{M^2}{\beta}\,\epsilon\left[\frac{4N-5}{2}\,f(x)+(N-1)\,x\,f'(x)\right]\right\} \tag{69}$$

Integrating for the moment according to equation (55b), replacing f by Y according to equation (1), and then extracting $c_{m_{\dot\alpha}}$ according to equation (B12) gives

$$c_{m_{\dot\alpha}}=\frac{8}{\beta^3}\left[\frac{1}{3}-\frac{1}{2}\,\frac{b}{c}-\frac{M^2(N-1)}{\beta}\,\frac{b}{c}\,\frac{1}{c^3}\int_0^c Y\,dx-\right.$$

$$\left.\frac{M^2}{2\beta}\,\frac{1}{c^3}\int_0^c(x-b)Y\,dx+\frac{M^2(N-1)}{\beta}\left(1-\frac{b}{c}\right)\frac{Y(c)}{c}\right] \tag{70}$$

The combination $(c_{m_q}+c_{m_{\dot\alpha}})$ is given, according to equation

(B12), by twice the coefficient of $(c\theta/U)$ in equations (58) and (60). Together with equation (70), this gives

$$c_{m_q}=\frac{8}{\beta}\left[-\frac{1}{3}+\frac{b}{c}\left(1-\frac{b}{c}\right)+\left(2\frac{M^2N-2}{\beta}+\frac{M^2}{2\beta^3}\right)\times\right.$$
$$\left.\frac{1}{c^3}\int_0^c(x-b)Ydx-\frac{M^2N-2}{\beta}\left(1-\frac{b}{c}\right)^2\frac{Y(c)}{c}\right] \quad (71)$$

These results have been derived independently by Martin and Gerber (refs. 15 and 16). The agreement serves as a further partial check on equation (67).

SECOND-ORDER SOLUTION FOR ARBITRARY FREQUENCIES

For some purposes the previous solution for slow oscillations may be inadequate. In principle, the second-order solution can be extended to include still higher powers of frequency, but the labor required is clearly prohibitive. Alternatively, one can attack directly the problem for arbitrary frequencies.

The second-order solution can, in fact, be carried out for a general airfoil at arbitrary frequencies. However, the result is formidable, involving multiple integrals of products of Bessel functions, and the reduction to simpler form for specific profiles appears to be difficult.

A more practical approach is to choose a specific airfoil shape in advance. Then the second-order solution involves only functions of the type encountered in the linearized theory. In particular, it is found that (at least for the simplest shape) the final expressions for lift and moment involve only functions which have been already studied and tabulated, so that numerical results are readily obtained.

MODIFIED SMOOTHING PROCEDURE

The smoothing discussed previously must be dropped at an earlier stage of the solution when a specific airfoil shape is chosen. It is therefore necessary first to modify the differential equation and boundary conditions so that no singular terms appear.

Consider first the differential equation (eq. (31)). Applying the Laplace transformation of equation (36), and envisioning the smoothed problem, so that ψ and ψ_x vanish at $x=0$, reduces it to

$$\bar{\Psi}_{yy}-\beta^2\left(s^2+\frac{\kappa^2}{M^2}\right)\bar{\Psi}=2(M^2s-i\kappa)\left[\beta^2(N-1)\mathcal{L}\{\varphi_x\psi_x\}+\right.$$
$$\left.\mathcal{L}\{\varphi_y\psi_y\}-i\kappa\frac{\beta^2N}{M^2}\mathcal{L}\{\varphi_x\psi\}\right] \quad (72)$$

In this form, all troublesome second derivatives have disappeared from the right-hand side, so that the smoothing can now be dropped.

Consider next the tangency condition of equation (25), which contains second derivatives of both φ and ψ. These cause no difficulty when, as in the solution for slow oscillations, the tangency condition is imposed as it stands. However, its Laplace transform will be used here, and then the Dirac delta functions associated with the second derivatives would affect the integration implied in the inversion of the transformation. These troublesome second derivatives can be eliminated by first expressing them in terms of x derivatives only through the first-order equations:

$$\left.\begin{aligned}\varphi_{yy}&=\beta^2\varphi_{xx}\\ \psi_{yy}&=\beta^2\left(\psi_{xx}+\frac{\kappa^2}{M^2}\psi\right)\end{aligned}\right\} \quad (73)$$

and then applying the Laplace transformation while envisioning the smoothed problem (so that φ, ψ, ψ_x, and φ_x vanish at $x=0$), which gives

$$\bar{\Psi}_y=-\bar{w}(s)-M^2\mathcal{L}\{e^{i\kappa x}\varphi_x\}+\epsilon M^2\mathcal{L}\{\psi_x f'\}-i\kappa\epsilon\mathcal{L}\{\psi f'\}-$$
$$\beta^2\epsilon s\mathcal{L}\{\psi_x f\}-\beta^2\epsilon\frac{\kappa^2}{M^2}\mathcal{L}\{\psi f\}+$$
$$\beta^2(s-i\kappa)\mathcal{L}\{e^{i\kappa x}\varphi_x(x-b)\} \qquad \text{at } y=0 \quad (74)$$

Again, all second derivatives have disappeared, so that the smoothing can be dropped.

Consider finally the upstream condition. For the smoothed problem, one statement of the condition was seen to be that the solution represents downgoing waves. This means that the complementary function for the iteration equation should have the same form as the linearized solution, and this statement of the upstream condition applies as well to the actual problem.

SOLUTION FOR WEDGE

The simple case of a single wedge illustrates the method of solution that can, in principle, be applied to any profile formed of piecewise analytic arcs. For a wedge of semivertex angle ϵ, the first-order solution for φ is given by equation (35) as

$$\varphi=-\frac{\epsilon}{\beta}(x-\beta y) \quad (75)$$

and the first-order solution for ψ is given by equation (42). Substituting into the right-hand side of equation (72) gives for the transformed iteration equation [7]

$$\bar{\Psi}_{yy}-\beta^2\left(s^2+\frac{\kappa^2}{M^2}\right)\bar{\Psi}=-2\epsilon(M^2s-i\kappa)\left[1+\right.$$
$$\left.\frac{(N-1)s-i\kappa N/M^2}{\sqrt{s^2+\kappa^2/M^2}}\right]\bar{w}(s)e^{-\beta y\sqrt{s^2+\kappa^2/M^2}} \quad (76)$$

A particular integral of this equation is given by an appropriate multiple of $ye^{-\beta y\sqrt{s^2+\kappa^2/M^2}}$, and adding a complementary function that represents downgoing waves gives the solution

$$\bar{\Psi}=C(s)e^{-\beta y\sqrt{s^2+\kappa^2/M^2}}+$$
$$\frac{\epsilon}{\beta}(M^2s-i\kappa)\left[\frac{1}{\sqrt{s^2+\kappa^2/M^2}}+\frac{(N-1)s-i\kappa N/M^2}{s^2+\kappa^2/M^2}\right]\bar{w}(s)ye^{-\beta y\sqrt{s^2+\kappa^2/M^2}} \quad (77)$$

The coefficient $C(s)$ is evaluated by imposing the tangency

[7] In this simple case, the partial particular integral of equation (33) offers little actual simplification.

condition of equation (74), with the result that the Laplace transform of the second-order solution is found to be

$$\overline{\Psi}=\frac{e^{-\beta y\sqrt{s^2+\kappa^2/M^2}}}{\beta\sqrt{s^2+\kappa^2/M^2}}\left\{\overline{w}(s)\left[1+\epsilon y(M^2s-i\kappa)+\right.\right.$$

$$\frac{\epsilon}{\beta}(M^2s-i\kappa)\frac{(N-1)s-i\kappa N/M^2}{\sqrt{s^2+\kappa^2/M^2}}\left(\beta y+\right.$$

$$\left.\left.\left.\frac{1}{\sqrt{s^2+\kappa^2/M^2}}\right)\right]-\beta\epsilon\sqrt{s^2+\kappa^2/M^2}\,\frac{d}{ds}\overline{w}(s)-\beta\epsilon b-\frac{\epsilon}{\beta}\frac{1}{s-i\kappa}\right\} \quad (78)$$

The inversion can be carried out using the standard tables (e. g., ref. 7) together with the convolution theorem. For calculating surface pressures, it suffices to obtain the solution in the plane $y=0$, which is found to be

$$\Psi(x,0)=\frac{1}{\beta}\int_0^x J_0\left(\frac{\kappa}{M}\xi\right)w(x-\xi)d\xi+\epsilon x\,w(x)+$$

$$\epsilon\frac{M^2(N-1)}{\beta^2}xw(0)J_0\left(\frac{\kappa}{M}x\right)-\epsilon b\,J_0\left(\frac{\kappa}{M}x\right)+$$

$$\frac{\epsilon}{\beta^2}\int_0^x J_0\left(\frac{\kappa}{M}\xi\right)\left\{\xi\left[M^2(N-1)w'(x-\xi)-\right.\right.$$

$$\left.\left.i\kappa(2N-1)w(x-\xi)-\kappa^2\frac{N}{M^2}\int_0^{x-\xi}w(\eta)d\eta\right]-e^{i\kappa(x-\xi)}\right\}d\xi \quad (79a)$$

where

$$w(x)=e^{i\kappa x}\left[1+i\kappa\frac{\beta^2}{M^2}(x-b)\right] \quad (79b)$$

With the pivot at the nose ($b=0$), this agrees with the result of applying linearized theory to the mean steady flow behind the shock wave. Also, when expanded in powers of frequency, it agrees with the previous low-frequency solution up to terms in κ^3.

The surface pressure coefficient can now be calculated from equation (27) and the lift and moment coefficients from equations (55).

EXAMPLE: WEDGE PIVOTED AT NOSE

For simplicity, the results will be given only for the special case of rotation about the nose. Then it is found that the lift and moment coefficients are given by

$$\left.\begin{matrix}\frac{c_l}{4\theta_0e^{i\omega t}}\\ \frac{-c_m}{4\theta_0e^{i\omega t}}\end{matrix}\right\}=\left(\frac{1}{\beta}+\epsilon\frac{M^2N-2}{\beta^2}\right)f_\nu+(2-\nu)i\lambda\left(\frac{1}{\beta}+\epsilon\right)(f_0-f_{1+\nu})-$$

$$\frac{\lambda^2}{2+\nu}\left(\frac{1}{\beta}+\frac{M^2}{\beta^2}\epsilon\right)(f_0-f_{2+\nu})+$$

$$\frac{\lambda^2}{1+\nu}\left[\frac{1}{\beta}+\epsilon\left(4\frac{M^2}{\beta^2}-\frac{3M^2+1}{M^2}N\right)\right](f_1-f_{2+\nu})-$$

$$\epsilon\frac{M^2}{\beta^2}\left(N-1+\frac{N}{M^6}\right)[(1+\nu)f_\nu-e^{-i\lambda M^2/\beta^2}J_0(M\lambda/\beta^2)]-$$

$$i\lambda\epsilon\left(3\frac{M^2}{\beta^2}-\frac{3M^4-M^2-1}{M^2\beta^2}N\right)f_{1+\nu}+$$

$$i\lambda^3\epsilon\left(\frac{M^2}{\beta^2}-N\right)\left[\frac{f_1}{2+\nu}-\frac{f_2}{1+\nu}+\frac{f_{3+\nu}}{(1+\nu)(2+\nu)}\right] \quad (80)$$

where $\nu=0$ for the lift and $\nu=1$ for the moment. The functions f_n, given by

$$f_n(M,\lambda)=\int_0^1 x^n e^{-i\lambda M^2x/\beta^2}J_0\left(\frac{M}{\beta^2}\lambda x\right)dx \quad (81)$$

arise in linearized theory for n ranging from 0 to 3. They have been studied and tabulated by von Borbély (ref. 18), Schwarz (ref. 19), and Garrick and Rubinow (ref. 11). They can all be expressed in terms of f_0 by a recurrence relation due to von Borbély (see, e. g., eq. (A.87) of ref. 12), so that the additional f_4 required here is easily computed.

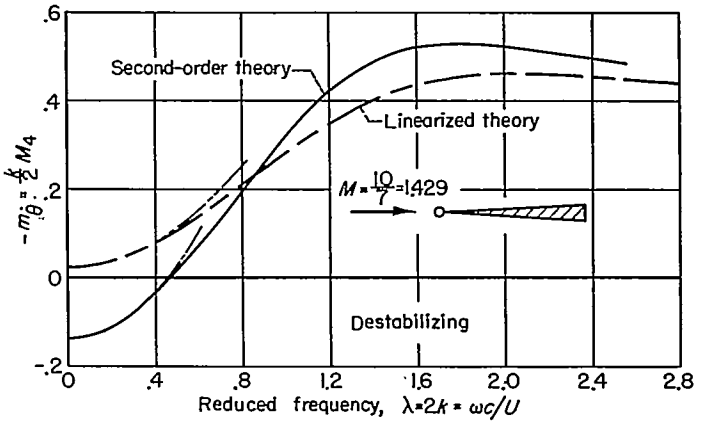


FIGURE 13.—Damping-moment coefficient for 10-percent-thick wedge oscillating about its vertex at $M=10/7$.

Figure 13 shows the variation of aerodynamic damping moment, according to first- and second-order theories, for a 10-percent-thick single wedge oscillating about its nose at a Mach number of 10/7. Also shown for comparison are the parabolic approximations of the low-frequency analysis. It is seen that the thickness effect is reversed at high flutter frequencies, as is suggested by the parabolic approximation.

DISCUSSION

HIGHER-ORDER EFFECTS

The moderate magnitude of second-order effects would suggest that the influence of third- and higher-order terms is of no practical importance, except perhaps in the transonic

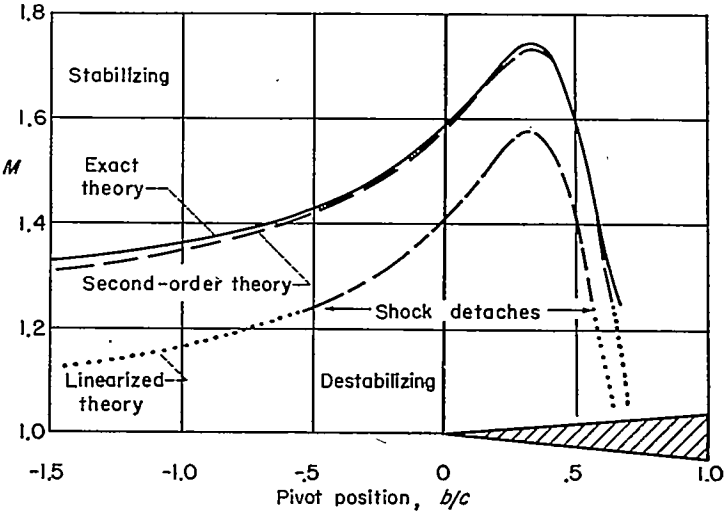


FIGURE 14.—Neutral damping boundary for wedge of 5° semivertex angle.

range near shock detachment. This supposition can be confirmed in the case of the single-wedge airfoil, for which a solution exact in thickness (but linearized with respect to angle of pitch) has been derived in references 2 and 3. Figure 14 compares the boundaries of neutral aerodynamic damping for a slowly oscillating wedge of 5° semivertex angle as predicted by the linearized, second-order, and exact theories. The second-order solution lies close to the exact result down to the Mach number for shock detachment (which is almost the same as the Mach number at which the flow ceases to be purely supersonic).[8]

APPLICATION TO FINITE-SPAN WINGS

Extension of the second-order solution to wings of finite aspect ratio does not seem possible at present. No second-order solution has yet been found even for steady flow past the simplest lifting wing.

Fortunately, the main conclusion to be drawn from the present analysis is that nonlinear thickness effects are quite moderate in magnitude. Practical supersonic wings will, therefore, probably be so thin that nonlinear effects are negligible, so that reliance can be placed in the predictions of linearized theory. Only if the wing is unduly thick, or if the Mach number is close to unity, or if unusual accuracy is required, may the engineer be forced to estimate the effects of thickness. In this event, he might assume that the effects of thickness are in some sense additive to those of aspect ratio, provided the aspect ratio is high and the frequency low. For example, the two-dimensional correction might be applied stripwise to the spanwise loading predicted by linearized theory. Some indication of the extent to which such an assumption would be justified can be obtained by considering other pairs of effects whose combined influence is known. Figure 12 shows that the effects of thickness and frequency are roughly additive for the frequencies of usual practical interest in dynamic stability analysis (say $\lambda<0.2$). Likewise, figure 15 shows that the effects of aspect ratio and frequency, determined from Watkins' linearized solution for the rectangular wing (ref. 20), are nearly additive in the same range of frequencies.

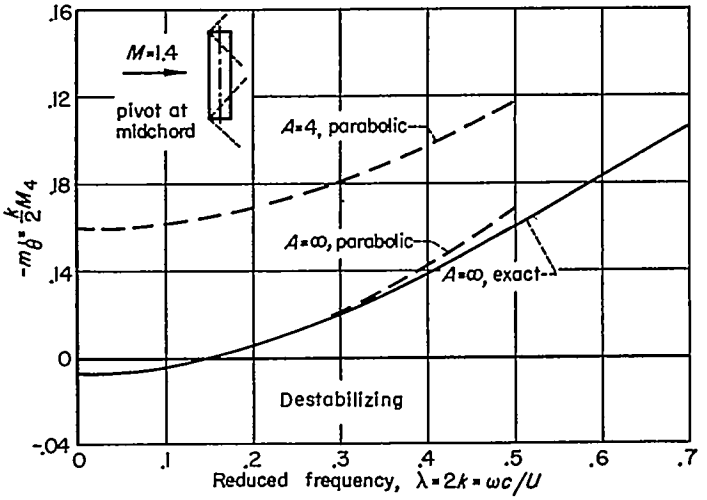


FIGURE 15.—Effect of aspect ratio upon damping-moment coefficient for rectangular wing according to linearized theory.

Martin and Gerber have calculated the second-order effects of thickness upon damping in roll for wings of infinite span (ref. 21). They then estimate the effect for a finite rectangular wing by increasing the result of linearized theory in the same ratio as for the infinite wing. Agreement with experiment is thus considerably improved, which gives further assurance that superposition of thickness effects may be justified also for oscillating wings.

Recently, Acum has estimated the aerodynamic damping of pitching oscillations of rectangular wings at supersonic speeds by assuming that the thickness effects of the present theory can be added to the aspect-ratio effects predicted by linearized theory (ref. 22).

FURTHER ANALYSIS

If thickness effects of the magnitude indicated by the present analysis are judged to be significant at flutter frequencies, extension of the high-frequency solution to more practical profiles would be warranted. The solution given for a single wedge at arbitrary frequency should be extended next to the biconvex or double-wedge airfoil. Although considerable computation is involved, it does not appear that the labor would be prohibitive.

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NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS
MOFFETT FIELD, CALIF., *Apr. 20, 1953*

[8] In figure 14 the curve designated second-order theory includes all second-order effects and also some (but not all) of the third-order effects. It therefore does not represent the second-order solution of this report. The curve for that solution (calculated from eqs. (58) and (60)) can be plotted from the following values:

b/c:	−1.5	−1.0	−0.5	0	0.33	0.5	0.6
λf:	1.21	1.26	1.34	1.52	1.63	1.52	1.26

The revised curve falls significantly below that shown for the exact theory. Hence the conclusion reached in the discussion of higher-order effects on pages 623 and 624 must be modified; third- and higher-order terms can be of some practical significance. The author is indebted to Dr. Georg Drougge for pointing out this discrepancy.

APPENDIX A

SYMBOLS

a	speed of sound
A	aspect ratio of wing
b	downstream distance from leading edge to pivot
c	airfoil chord
$C(s)$	constant of integration
c_l	section lift coefficient
c_m	section moment coefficient
c_{m_α}, c_{m_q}, $c_{m_{\dot\alpha}}$	stability derivatives (See eq. (B5).)
c_{m_θ}, $c_{m_{\dot h}}$, $c_{m_{\dot\theta}}$, $c_{m_{\ddot h}}$	flutter derivatives (See eq. (B6).)
C_p	pressure coefficient
C_{p_0}	surface pressure coefficient in mean steady flow
C_{p_u}, C_{p_l}	pressure coefficients on upper and lower surfaces of airfoil, respectively
$f(x)$	function defining upper surface of airfoil at zero angle of pitch
f_n	See equation (81).
$g(x)$	function defining amplitude of oscillation of upper surface of smooth airfoil
$h(t)$	elevation of plunging airfoil, positive downward
h_0	amplitude of harmonic plunging oscillation
k	reduced frequency, $\frac{\omega c}{2U}=\frac{1}{2}\lambda$
m_h, m_θ, $m_{\dot h}$, $m_{\dot\theta}$	British flutter coefficients (See eq. (B9).)
M	free-stream Mach number
M_1, M_2, M_3, M_4	American flutter coefficients (See eq. (B8).)
N	$\frac{\gamma+1}{2}\frac{M^2}{\beta^2}$
p	static pressure
q	rate of pitching, $\dot\theta$
s	Laplace transformation variable
$S(x, y, t)$	function defining moving surface
t	time
u, v	velocity components parallel and perpendicular to free stream
U	speed of flight
$\vec{V}$	velocity vector
$w(x)$	downwash velocity at $y=0$
x, y	coordinates parallel and perpendicular to flight direction, moving with mean steady velocity of airfoil
$Y(x)$	ordinate of upper surface of airfoil at zero angle of pitch
z	$x-\beta y$
α	angle of attack—angle of airfoil with respect to relative wind at pivot
β	$\sqrt{M^2-1}$
γ	adiabatic exponent of gas
Δ	Laplacian operator, $\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$
ϵ	small parameter representative of airfoil thickness
θ	angle of pitch—angle of airfoil with respect to x axis
θ_0	angular amplitude of harmonic pitching oscillation
κ	$\frac{M^2\omega}{\beta^2 U}$
λ	reduced frequency, $\frac{\omega c}{U}=2k$
ν	constant that is 0 or 1 in equation (80)
ρ	density
τ	airfoil thickness ratio
φ	first-order mean steady perturbation potential
ϕ	second-order mean steady perturbation potential
Φ	complete perturbation potential
ψ	first-order time-dependent perturbation potential
Ψ	second-order time-dependent perturbation potential
Ψ_*	partial particular integral of time-dependent iteration equation
ω	angular frequency of oscillation
Ω	complete velocity potential
$(\,)_0$	free-stream conditions
$(\,)_1$	conditions in mean steady flow behind bow shock wave on wedge
$(\dot{\cdot})$	differentiation with respect to time
$\overline{\ }$, $\mathcal{L}\{\ \}$	Laplace transform

APPENDIX B

CONNECTION BETWEEN FLUTTER AND STABILITY DERIVATIVES

In flutter analysis the motion of the airfoil is described by [1] the angle of pitch θ and the (negative) elevation h, whereas in stability analysis it is described by the angle of attack α and the rate of pitching q. From figure 16 it is seen that

$$\tan \alpha = \frac{\tan \theta + \dot{h}/U}{1 - (\dot{h}/U) \tan \theta} \tag{B1}$$

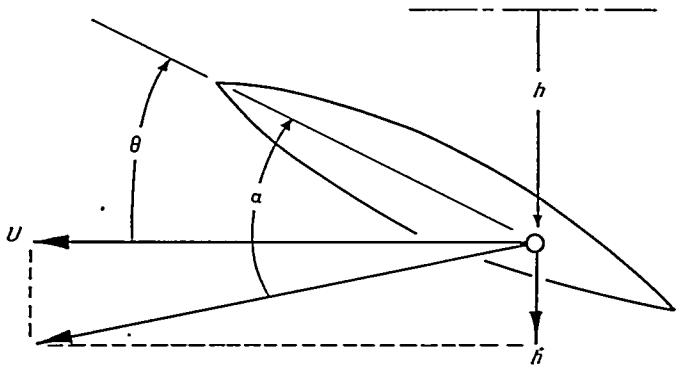

FIGURE 16.—Alternative coordinates for describing motion of oscillating airfoil through still air.

and by definition $q = \dot{\theta}$. Hence to second order in the angles, these alternative coordinate systems are related by

$$\left.\begin{aligned} \alpha &= \theta + \frac{\dot{h}}{U} \\ \alpha &= \dot{\theta} \end{aligned}\right\} \tag{B2}$$

For a given profile, gas, Mach number, and pivot, the moment coefficient at any instant depends on the entire previous history of the airfoil motion. Thus in stability coordinates

$$C_m(t) = C_m[\![\alpha(t), q(t)]\!] \tag{B3}$$

Here the heavy brackets are used to emphasize that this is not an ordinary function, but a functional of the two coordinates, which are themselves functions of past time. Now if α and q are analytic functions (for all past time), they can be resolved into their successive time derivatives. Then c_m assumes the nature of an ordinary function of these infinitely many variables, each of which depends upon time as a parameter:

$$c_m(t) = c_m[\alpha(t), \dot{\alpha}(t), \ldots, q(t), \dot{q}(t), \ldots] \tag{B4}$$

Finally, if c_m is analytic in each of these variables, Taylor series expansion gives [2]

$$c_m(t) = \alpha(t) c_{m_\alpha} + \left(\frac{c}{2U}\right) q(t) c_{m_q} + \left(\frac{c}{2U}\right) \dot{\alpha}(t) c_{m_{\dot{\alpha}}} + \ldots \tag{B5}$$

where only linear terms have been retained. Here the coefficients have been made dimensionless using the characteristic time $c/2U$, in accordance with the usual American notation. In stability analysis the motion is usually assumed so slow and smooth that the three terms shown explicitly are sufficient. The coefficients c_{m_α}, c_{m_q}, $c_{m_{\dot{\alpha}}}$, etc., which are (aside from factors of $c/2U$) the first partial derivatives of the function c_m, are the stability derivatives.

Proceeding similarly in flutter coordinates would lead to a corresponding expansion in terms of flutter derivatives:

$$c_m(t) = \theta(t) c_{m_\theta} + \frac{1}{U} \dot{h}(t) c_{m_{\dot{h}}} + \left(\frac{c}{2U}\right) \dot{\theta}(t) c_{m_{\dot{\theta}}} + \frac{1}{U}\left(\frac{c}{2U}\right) \ddot{h}(t) c_{m_{\ddot{h}}} + \ldots \tag{B6}$$

Here the first term has been omitted since it is clear physically that c_{m_h} vanishes identically—the moment depends on changes in elevation but not on the elevation itself. However, it is customary in flutter analysis to consider only harmonic oscillations:

$$\theta(t) = \theta_0 e^{i\omega t}, \quad h(t) = h_0 e^{i\omega t} \tag{B7}$$

and then the infinite series for c_m collapses to just four terms. In the usual American notation for flutter coefficients (ref. 11)

$$c_m = -2k^2 \left(2\frac{h}{c} M_1 + \frac{\dot{h}}{kU} M_2 + \theta M_3 + \frac{c\dot{\theta}}{2kU} M_4\right) \tag{B8}$$

and in the usual British notation (refs. 12, 13), with (θ, h) in place of (α, z)

$$c_m = 2\left(\frac{h}{c} m_h + \frac{\dot{h}}{U} m_{\dot{h}} + \theta m_\theta + \frac{c\dot{\theta}}{U} m_{\dot{\theta}}\right) \tag{B9}$$

Comparing these expressions with equation (B6) in the special case of harmonic oscillations shows that

$$\left.\begin{aligned} -2k^2 M_1 &= m_h = -k^2 c_{m_{\ddot{h}}} + \ldots \\ -2k\, M_2 &= 2m_{\dot{h}} = c_{m_{\dot{h}}} - k^2 c_{m_{\ddot{\theta}}} + \ldots \\ -2k^2 M_3 &= 2m_\theta = c_{m_\theta} - k^2 c_{m_{\ddot{\theta}}} + \ldots \\ -2k\, M_4 &= 4m_{\dot{\theta}} = c_{m_{\dot{\theta}}} - k^2 c_{m_{\ddot{\theta}}} + \ldots \end{aligned}\right\} \tag{B10}$$

The relations between derivatives in the two coordinate systems are found by regarding each system as consisting of an infinite number of coordinates that are related by equation (B2) together with all the relations obtained by

[1] British writers use z instead of h. Both American and British writers have previously used α instead of θ, but θ is preferable for the reason given in footnote 2, and is apparently now being adopted by the British.

[2] The reader may choose to regard this result as being obvious from physical considerations, in which case the preceding mathematical formalism is superfluous.

differentiating equation (B2) repeatedly with respect to time. Hence

$$\left.\begin{array}{ll} c_{m_\theta}=c_{m_\alpha} & c_{m_\alpha}=c_{m_{\dot h}} \\ c_{m_{\dot h}}=c_{m_\alpha} & c_{m_q}=c_{m_{\dot\theta}}-c_{m_{\ddot h}} \\ c_{m_{\dot\theta}}=c_{m_q}+c_{m_{\dot\alpha}} & c_{m_{\dot\alpha}}=c_{m_{\ddot h}} \\ c_{m_{\ddot h}}=c_{m_{\dot\alpha}} & \cdots\cdots \\ \cdots\cdots & \end{array}\right\} \tag{B11}$$

Finally, combining these relations with those of equation (B9) gives the omnibus relations

$$\left.\begin{array}{l} -2k^2M_1=\ m_{\dot h}=-k^2c_{m_{\ddot h}}+\ \ldots\ =-k^2c_{m_{\dot\alpha}}+\ \ldots \\ -2k\,M_2=2m_{\dot h}=c_{m_{\dot h}}-\ \ldots\ \ =c_{m_\alpha}-\ \ldots \\ -2k^2M_3=2m_\theta=c_{m_\theta}-\ \ldots\ \ =c_{m_\alpha}-\ \ldots \\ -2k\,M_4=4m_{\dot\theta}=c_{m_{\dot\theta}}-\ \ldots\ \ =(c_{m_q}+c_{m_{\dot\alpha}})-\ \ldots \end{array}\right\} \tag{B12}$$

The corresponding relations for lift can be found in the same way.

Equation (B12) shows that the combination $(c_{m_q}+c_{m_{\dot\alpha}})$ is proportional to the first term in the expansion of the flutter coefficient $m_{\dot\theta}$ or M_4 in powers of frequency. It is therefore given theoretically by the solution linear in frequency, and experimentally by measurements at low frequency. However, to determine $c_{m_{\dot\alpha}}$ alone, and hence c_{m_q}, squares of frequency must be retained. For this reason, it is difficult to find c_{m_q} and $c_{m_{\dot\alpha}}$ by oscillating an airfoil in a conventional wind tunnel. It would be necessary to perform the experiment at various frequencies and so to determine the initially parabolic variation with frequency of the component of moment that is out of phase with the angle of pitch.

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